Econometrics II. Lecture Notes 2

SIMULTANEOUS LINEAR EQUATIONS SYSTEMS

1. Identification
2. Estimation
3. Identification with cross-equation
4. Identification with covariance restrictions
5. Models nonlinear in the endogenous variables

2.1 Identification

Estimation was treated in the previous classes, which covered also the omitted variables and measurement error problems.

• In some cases equations in SEMs have a proper separate economic meaning (demand, supply equations) [autonomous] and generally they describe a causal relationship.

• In other applications the endogenous variables in a SEM are not autonomous since they are all choice variables of the same economic unit. In this case endogenous variables appear as regressors in other equations and do not describe any structural relationship: structural and simultaneous are different concepts.

2.1.1 Exclusion restrictions. Reduced forms

Consider the **structural equations** for the endogenous variables $y_1, \ldots, y_G$ written as a linear SEM for a population

$$\begin{aligned} y_1 &= \mathbf{y}'_{(1)}\boldsymbol{\gamma}_{(1)} + \mathbf{z}'_{(1)}\boldsymbol{\delta}_{(1)} + u_1 \\ &\vdots \\ y_G &= \mathbf{y}'_{(G)}\boldsymbol{\gamma}_{(G)} + \mathbf{z}'_{(G)}\boldsymbol{\delta}_{(G)} + u_G \end{aligned} \tag{2.1}$$

where for each equation g, $\mathbf{y}_{(g)}$ is $G_g \times 1$ and $\mathbf{z}_{(g)}$ is $M_g \times 1$. (Equilibrium conditions imposed.)

The $\mathbf{y}_{(g)}$ vector denotes the endogenous variables that appear on the right hand side of the g-equation: it may contain any variable y_h, $h \neq g$. $\mathbf{z}_{(g)}$ are the exogenous variables appearing in equation g. There can overlap among the different equations.

■ The restrictions imposed in (3.1) are **exclusion restrictions**, because some endogenous and exogenous variables are excluded from some equations.

We assume that

$$\mathbb{E}\left[\mathbf{z}u_g\right] = \mathbf{0}, \quad g = 1, \ldots, G, \tag{2.2}$$

where $\mathbf{z}$ is the vector of all M (non repeated) exogenous variables. Under a true structural model we can assume instead that

$$\mathbb{E}\left[u_g|\mathbf{z}\right] = \mathbf{0}, \quad g = 1, \ldots, G,$$

but this is not necessary (and would not hold for e.g. measurement error models).

Since exogenous variables appearing in any equation are orthogonal to any error, elements of $\mathbf{z}$ not appearing in a particular equation do not have any role in this particular structural relationship. If there are not (exclusion) restrictions available is because this equation is not truly autonomous.

We also assume that

$$\mathbb{E}[\mathbf{z}\mathbf{z}'] \text{ is non singular.}$$

Since generally u_g is correlated with $\mathbf{y}_{(g)}$, OLS and GLS are inconsistent: IV methods.

■ Other identification information: restrictions on covariance matrix of $\mathbf{u}$, $\mathbf{\Sigma} := \mathbb{V}[\mathbf{u}]$.

EX. 1 (SEM: Market Equilibrium, Working (1927)) *Demand and Supply Model:*

$$\begin{aligned} q^d &= \alpha_0 + \alpha_1 p + u \quad \text{(Demand equation)} \\ q^s &= \beta_0 + \beta_1 p + v \quad \text{(Supply equation)} \\ q^d &= q^s \quad \text{(market equilibrium)} \end{aligned}$$

where q^d is the quantity demanded for the commodity in question, q^s is the quantity supplied and p is the price.

The error term in the demand eq., u, represents factors that influence demand other than price, which can make the demand curve to shift up or down.

The error term v, represents factors that influence supply other than price.

■ We can assume that

$$\mathbb{E}(u) = 0, \ \ \mathbb{E}(v) = 0, \ \ \mathbb{C}(u, v) = 0.$$

If we set $q = q^d = q^s$ the system reduces to

$$\begin{aligned} q &= \alpha_0 + \alpha_1 p + u \quad \text{(Demand equation)} \\ q &= \beta_0 + \beta_1 p + v \quad \text{(Supply equation)} \end{aligned}$$

If the equation has an intercept, then the orthogonality condition, $\mathbb{E}(xe) = 0$, is violated if and only if the regressor is correlated with the current error term, $\mathbb{C}(x, e) \neq 0$.

■ **Reduced form.** The regressor p is necessarily endogenous in both equations, because solving for (p, q)

$$\begin{aligned} p &= \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} + \frac{v - u}{\alpha_1 - \beta_1} \\ q &= \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1} + \frac{\alpha_1 v - \beta_1 u}{\alpha_1 - \beta_1} \end{aligned}$$

so price is a function of the two error terms and

$$\mathbb{C}(p, u) = -\frac{\mathbb{V}(u)}{\alpha_1 - \beta_1} \qquad \mathbb{C}(p, v) = \frac{\mathbb{V}(v)}{\alpha_1 - \beta_1},$$

which are nonzero, $\sigma_v^2 > 0$, $\sigma_u^2 > 0$.

■ **OLS estimation:**

$$q = \gamma_0 + \gamma_1 p + \epsilon$$

which is $\hat{\gamma}_{n1}$ estimating? (which curve, D or S?).

The least squares estimate should estimate the projection of q on $(1,\ p)$, that is,

$$\hat{\gamma}_{n1} \rightarrow_p \frac{\mathbb{C}(p, q)}{\mathbb{V}(p)}$$

so using the demand curve ($q^d = \alpha_1 + \alpha_2 p + u$), we have that

$$\mathbb{C}(p, q) = \alpha_1 \mathbb{V}(p) + \mathbb{V}(p, u)$$

and

$$\hat{\gamma}_{n1} \rightarrow_p \alpha_1 + \frac{\mathbb{C}(p, u)}{\mathbb{V}(p)}$$

■ Using the supply equation ($q^s = \alpha_1 + \alpha_2 p + u$) we obtain that

$$\hat{\gamma}_{n1} \rightarrow_p \beta_1 + \frac{\mathbb{C}(p, v)}{\mathbb{V}(p)}.$$

As both covariances terms are not 0, the OLS estimate is consistent for neither α_1 or β_1. These factors are the **endogeneity bias**, or **simultaneous equations bias** or **simultaneous bias**.

If there were no shifts in demand, $u = 0$, then $\mathbb{C}(p, u) = 0$ and the OLS estimate would be consistent for the demand parameter, α_1, and the same holds for β_1 when $v = 0$. If both curves shift then the OLS estimates a weighted average of both coefficients,

$$\hat{\gamma}_{n1} \rightarrow_p \frac{\alpha_1 \mathbb{V}(v) + \beta_1 \mathbb{V}(u)}{\mathbb{V}(v) + \mathbb{V}(u)}. \tag{2.3}$$

■ The problem is that when both curves can shift we cannot infer from data whether the change in price and quantity is due to a demand shift or a supply shift, but things may change if we could have additional information of some factors that shift the supply (demand) curve.

Suppose that the supply shifter, v, can be split into two factors, an observable z and an unobservable ζ, uncorrelated with z, so

$$q = \beta_0 + \beta_1 p + \underbrace{\beta_2 z + \zeta}_{v}, \quad \text{with} \quad \beta_2 \neq 0. \quad \text{(supply equation)}$$

Suppose that z is exogenous in the demand equation (i.e. uncorrelated with u) : then can extract which price movements are associated with z (Supply) but uncorrelated with u in the Demand equation, making possible the estimation of the demand curve examining the relationship between consumption q and z.

⇒ The **D equation is identified** if there is one element z in the S equation which is not in the D equation.

■ If we solve the D-S system we get now

$$\begin{aligned} p &= \frac{\beta_0 - \alpha_0}{\alpha_1 - \beta_1} + \frac{\beta_2}{\alpha_1 - \beta_1} z + \frac{\zeta - u}{\alpha_1 - \beta_1} \\ q &= \frac{\alpha_1 \beta_0 - \alpha_0 \beta_1}{\alpha_1 - \beta_1} + \frac{\alpha_1 \beta_2}{\alpha_1 - \beta_1} z + \frac{\alpha_1 \zeta - \beta_1 u}{\alpha_1 - \beta_1}. \end{aligned}$$

Since $\mathbb{C}(z, \zeta) = 0$ by construction and $\mathbb{C}(z, u) = 0$ by assumption, it follows that

$$\mathbb{C}(z, p) = \frac{\beta_2}{\alpha_1 - \beta_1} \mathbb{V}(z) \neq 0.$$

With a valid instrument then we can estimate the coefficient α_1 in D, because

$$\begin{aligned} \mathbb{C}(z, q) &= \alpha_1 \mathbb{C}(z, p) + \mathbb{C}(z, u) \\ &= \alpha_1 \mathbb{C}(z, p) \end{aligned}$$

so

$$\alpha_1 = \frac{\mathbb{C}(z, q)}{\mathbb{C}(z, p)},$$

and a natural estimate is the **Instrumental Variables (IV) estimate**:

$$\hat{\alpha}_{n1}^{IV} = \frac{\mathbb{C}_n(z, q)}{\mathbb{C}_n(z, p)}.$$

■ Another estimate is the **Two-Stage Least Squares (2SLS) estimate:** it is obtained in two successive regressions:

1. $p|1,\ z \quad \Rightarrow \quad \hat{p}_n = \mathbb{L}_n(p|z)$.

2. $q|1,\ \hat{p}_n \quad \Rightarrow \quad \hat{\alpha}_{n1}^{2SLS}$.

The use of $\hat{p}_n$ in the second regression is the difference wrt OLS, so

$$\hat{\alpha}_{n1}^{2SLS} = \frac{\mathbb{C}_n(\hat{p}_n, q)}{\mathbb{V}_n(\hat{p})} = \frac{\mathbb{C}_n(\mathbb{L}_n(p|z), q)}{\mathbb{V}_n(\mathbb{L}_n(p|z))},$$

so in fact,

$$q = \alpha_0 + \alpha_1 \hat{p}_n + \underbrace{\{u + \alpha_1 (p - \hat{p}_n)\}}_{\text{error term of reg. 2.}}.$$

□

■ We consider the **identification of the first equation** in system (3.1),

$$\begin{aligned} y_1 &= \mathbf{y}'_{(1)}\boldsymbol{\gamma}_{(1)} + \mathbf{z}'_{(1)}\boldsymbol{\delta}_{(1)} + u_1 \\ &= \mathbf{x}'_{(1)}\boldsymbol{\beta}_{(1)} + u_1, \qquad \mathbf{x}'_{(1)} := \left(\mathbf{y}'_{(1)}, \mathbf{z}'_{(1)}\right) \end{aligned}$$

where $\mathbf{x}_{(1)}$ is $K_1 \times 1$, $K_1 := G_1 + M_1$, which is similar to the general linear system studied previously. Only *exclusion* restrictions are imposed.

■ The **reduced form** for $\mathbf{y}_{(1)}$ (which is $G_1 \times 1$)

$$\mathbf{y}_{(1)} = \boldsymbol{\Pi}'_{(1)}\mathbf{z} + \mathbf{v}_{(1)}$$

where $\mathbb{E}\left[\mathbf{z}\mathbf{v}'_{(1)}\right] = \mathbf{0}$ and $\boldsymbol{\Pi}_{(1)}$ is $M \times G_1$.

Let $\mathbf{S}_{(1)}$ be an $M_1 \times M$ selection matrix, such that

$$\mathbf{z}_{(1)} = \mathbf{S}_{(1)}\mathbf{z}.$$

■ The **rank condition** is

$$\text{rank}\,\mathbb{E}\left[\mathbf{z}\mathbf{x}'_{(1)}\right] = K_1, \qquad K_1 := G_1 + M_1. \tag{2.4}$$

Then

$$\mathbb{E}\left[\mathbf{z}\mathbf{x}'_{(1)}\right] = \mathbb{E}\left[\mathbf{z}\left(\mathbf{z}'\boldsymbol{\Pi}_{(1)}, \mathbf{z}'\mathbf{S}'_{(1)}\right)\right] = \mathbb{E}\left[\mathbf{z}\mathbf{z}'\right] \begin{bmatrix} \underset{M\times G_1}{\boldsymbol{\Pi}_{(1)}} & \underset{M\times M_1}{\mathbf{S}'_{(1)}} \end{bmatrix}.$$

Since $\mathbb{E}\left[\mathbf{z}\mathbf{z}'\right]$ is full rank, the rank condition (3.4) is the same as

$$\text{rank}\ \left[\boldsymbol{\Pi}_{(1)}\ \mathbf{S}'_{(1)}\right] = G_1 + M_1 \equiv K_1,$$

i.e. $\left[\boldsymbol{\Pi}_{(1)}\ \mathbf{S}'_{(1)}\right]$ is full column rank.

■ Since the $\left[\boldsymbol{\Pi}_{(1)}\ \mathbf{S}'_{(1)}\right]$ matrix is $M \times (G_1 + M_1)$, we obtain the necessary **order condition**:

$$M \geq G_1 + M_1 \quad \Leftrightarrow \quad M - M_1 \geq G_1. \tag{2.5}$$

This states that the number of exogenous variables not appearing in the first equation, $M - M_1$, have to be at least as large as the number of endogenous variables appearing on the right hand side of the first equation, G_1.

2.1.2 General linear restrictions

The previous analysis is useful when reduced equations are available as well as the structural equations. Now we study identification in terms of the **structural parameters**:

$$\begin{aligned}\mathbf{y}'\boldsymbol{\gamma}_1 + \mathbf{z}'\boldsymbol{\delta}_1 + u_1 &= 0 \\ &\vdots \\ \mathbf{y}'\boldsymbol{\gamma}_G + \mathbf{z}'\boldsymbol{\delta}_G + u_G &= 0\end{aligned} \tag{2.6}$$

where $\mathbf{y}$ is $G \times 1$ vector of all endogenous variables and $\mathbf{z}$ is $M \times 1$ vector of all exogenous variables (and usually the intercept). Here the $\boldsymbol{\gamma}_g$ are $G \times 1$ and $\boldsymbol{\delta}_g$ are $M \times 1$ for all $g = 1, \ldots, G$: they are **unrestricted** (structural) parameters. We assume that (3.2) holds and $\mathbb{E}[\mathbf{z}\mathbf{z}']$ is full rank.

We can write the system as

$$\boldsymbol{\Gamma}'\mathbf{y} + \boldsymbol{\Delta}'\mathbf{z} + \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\Sigma} := \mathbb{E}[\mathbf{u}\mathbf{u}'] \tag{2.7}$$

where $\boldsymbol{\Gamma} := [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_G]$ is $G \times G$ and $\Delta := [\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_G]$ is $M \times G$, $\mathbf{u}$ is $G \times 1$.

■ The **reduced form** is

$$\begin{aligned}\mathbf{y} &= \left(-\boldsymbol{\Gamma}'^{-1}\boldsymbol{\Delta}'\right)\mathbf{z} + \left(-\boldsymbol{\Gamma}'^{-1}\right)\mathbf{u} = \boldsymbol{\Pi}'\mathbf{z} + \mathbf{v}, \\ \boldsymbol{\Lambda} &:= \mathbb{E}[\mathbf{v}\mathbf{v}'] = \boldsymbol{\Gamma}'^{-1}\boldsymbol{\Sigma}\boldsymbol{\Gamma}'^{-1\prime}\end{aligned}$$

where $\boldsymbol{\Pi} := -\boldsymbol{\Delta}\boldsymbol{\Gamma}'^{-1\prime} = -\boldsymbol{\Delta}\boldsymbol{\Gamma}^{-1}$ and $\mathbf{v} := \left(-\boldsymbol{\Gamma}'^{-1}\right)\mathbf{u}$.

Because $\mathbb{E}[\mathbf{z}\mathbf{v}'] = \mathbf{0}$ and $\mathbb{E}[\mathbf{z}\mathbf{z}']$ is nonsingular, $\boldsymbol{\Pi}$ and $\boldsymbol{\Lambda}$ are identified because they can be estimated by OLS given the reduced form and data $\mathbf{y}, \mathbf{z}$.

■ *When can we recover the structural parameters* $\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}$, $\boldsymbol{\Sigma}$ *from the reduced form parameters*? Without some restrictions this would be imposible.

Let $\mathbf{F}$ be a nonsingular $G \times G$ matrix. Then if we premultiply (3.7) by $\mathbf{F}$ we obtain

$$\begin{aligned}\mathbf{F}\boldsymbol{\Gamma}'\mathbf{y} + \mathbf{F}\boldsymbol{\Delta}'\mathbf{z} + \boldsymbol{\Delta}\mathbf{u} &= \mathbf{0} \\ \boldsymbol{\Gamma}^{*\prime}\mathbf{y} + \boldsymbol{\Delta}^{*\prime}\mathbf{z} + \mathbf{u}^* &= \mathbf{0}.\end{aligned}$$

However the reduced form for this system is identically the same as before: they are observationally equivalent and the structural parameters are not identified.

■ Let denote the **structural parameters** as

$$\underset{(G+M)\times G}{\mathbf{B}} := \begin{pmatrix} \mathbf{\Gamma} \\ \mathbf{\Delta} \end{pmatrix}, \qquad \mathbf{B} = [\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_G], \qquad \underset{(G+M)\times 1}{\boldsymbol{\beta}_g} = \begin{pmatrix} \boldsymbol{\gamma}_g \\ \boldsymbol{\delta}_g \end{pmatrix}.$$

$\mathbf{F}$ represents an **admisible linear transformation** if

1. $\mathbf{F}\mathbf{B}'$ satisfies all restrictions on $\mathbf{B}'$.
2. $\mathbf{F}\mathbf{\Sigma}\mathbf{F}'$ satisfies all restrictions on $\mathbf{\Sigma}$.

The system is identified if $\mathbf{F} = \mathbf{I}$ is the only admisible linear transformation. The restrictions for this usually rely on $\mathbf{B}$, but also $\mathbf{\Sigma}$ can be used.

■ For the first equation

$$\mathbf{y}'\boldsymbol{\gamma}_1 + \mathbf{z}'\boldsymbol{\delta}_1 + u_1 = 0, \qquad \underset{(G+M)\times 1}{\boldsymbol{\beta}_1} = \begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\delta}_1 \end{pmatrix}$$

or

$$\gamma_{11}y_1 + \gamma_{12}y_2 + \cdots + \gamma_{1G}y_G + z_1\delta_{11} + z_2\delta_{12} + \cdots + z_M\delta_{1M} = 0.$$

The first restriction is a **normalization restriction**, so one element in $\boldsymbol{\gamma}_1$ is -1 : one variable is then the lhs explained variable. Usually there is a clear choice in practice. With this normalization there are $G + M - 1$ unknown parameters on $\boldsymbol{\beta}_1$.

■ Suppose that the information on $\boldsymbol{\beta}_1$ is given via a set of **homogeneous linear restrictions**:

$$\underset{J_1\times(G+M)}{\mathbf{R}_1} \boldsymbol{\beta}_1 = \mathbf{0}$$

where J_1 is the number of known (additional) restrictions on $\boldsymbol{\beta}_1$ and is also the rank of $\mathbf{R}_1$ (no redundant restrictions).

With the normalization, these allow also for nonhomogenous restrictions (since e.g. the first element of $\boldsymbol{\beta}_1$ is -1).

■ *When are these restrictions and the normalization enough to identify* $\boldsymbol{\beta}_1$?

Since the transformed parameters are $\mathbf{B}^{*\prime} := \mathbf{F}\mathbf{B}'$, or $\mathbf{B}^* := \mathbf{B}\mathbf{F}'$ denote the first column of $\mathbf{B}^*$ by $\boldsymbol{\beta}_1^* = \mathbf{B}\mathbf{f}_1$ where $\mathbf{f}_1$ is the first column of $\mathbf{F}'$. Then $\boldsymbol{\beta}_1^*$ satisfies the restrictions iff

$$\mathbf{R}_1\boldsymbol{\beta}_1^* = \mathbf{R}_1(\mathbf{B}\mathbf{f}_1) = (\mathbf{R}_1\mathbf{B})\,\mathbf{f}_1 = \mathbf{0}. \tag{2.8}$$

• This is true if $\mathbf{f}_1 = \mathbf{e}_1 := (1, 0, \ldots, 0)'$ (or any scalar transformation) since then $\boldsymbol{\beta}_1^* = \mathbf{B}\mathbf{f}_1 = \boldsymbol{\beta}_1$. These should be the only vectors satisfying the condition (3.8).

• This implies that the null space of $\mathbf{R}_1\mathbf{B}$ should have dimension one, i.e.

$$\operatorname{rank}[\mathbf{R}_1\mathbf{B}] = G - 1 \tag{2.9}$$

since $\mathbf{R}_1\mathbf{B}$ has G columns. This is the **rank condition** for identification of $\boldsymbol{\beta}_1$.

■ $\mathbf{R}_1\mathbf{B}$ depends on the structural parameters of all equations:

$$\underset{J_1\times G}{\mathbf{R}_1\mathbf{B}} = \left[\underbrace{\mathbf{R}_1\boldsymbol{\beta}_1}_{=0},\ \mathbf{R}_1\boldsymbol{\beta}_2,\ \ldots,\ \mathbf{R}_1\boldsymbol{\beta}_G\right].$$

Since $\mathbf{R}_1\boldsymbol{\beta}_1 = \mathbf{0}$, the first column of $\mathbf{R}_1\mathbf{B}$ is $\mathbf{0}$: $\mathbf{R}_1\mathbf{B}$ can not have rank larger than $G-1$, so the other set of $G-1$ columns should be full rank.

Since $\boldsymbol{\Gamma}$ is nonsingular, $\mathbf{B}$ necessarily has rank G (full column rank). Therefore we must have that

$$\operatorname{rank}[\mathbf{R}_1] = J_1 \geq G - 1.$$

This is the, only necessary, **order condition.** This may hold, but the rank condition may fail.

■ When the restrictions on $\boldsymbol{\beta}_1$ consist **only on normalization and exclusion restrictions**, then both order conditions (on reduced and structural forms) are the same.

In this case $\mathbf{R}_1$ consists only of zeros and ones, and the number of rows of $\mathbf{R}_1$ is equal to the number of excluded endogenous variables, $G - G_1 - 1$, plus the number of excluded exogenous variables, $M - M_1$:

$$\underbrace{(G - G_1 - 1) + (M - M_1)}_{=J_1} \geq G - 1,$$

which is equal to (3.5),

$$M - M_1 \geq G_1.$$

■ We say that the first equation is

- **Unindentified**: if the rank condition (3.9) fails.
- **Just identified**: if the rank condition (3.9) holds, and $J_1 = G - 1$.
- **Over-identified**: if the rank condition (3.9) holds, and $J_1 > G - 1$: generally if one restriction is dropped, the equation remains identified.

Number of overidentifying restrictions: $J_1 - (G - 1)$.

2.2 Estimation

SEMs with linear homogenous restrictions can be rewritten as general systems of the form (3.1) only with exclusion restrictions if some variables are redefined.

These **structural** systems can be estimated using the techniques already studied: either system or equation by equation techniques if rank conditions are satisfied, such as 3SLS and chi-square estimates. Also the more robust equation by equation 2SLS can be used.

Efficiency improvements with system estimation are possible only if $\hat{\mathbf{\Sigma}}$ is not diagonal and not all equations are just identified.

Generally structural parameters are more interesting because:

- they have immediate economic meaning,
- allow study of political interventions,
- and can produce more efficient estimates of reduced form parameters.

■ In other situations we might be interested in estimating directly the **reduced form** parameters since this allows proper analysis of the effects of a modification of a exogenous variable.

Reduced form equations can be estimated by means of OLS: OLS equation by equation is equal to SUR (FGLS) estimation in absence of restrictions.

Robustness-Efficiency trade-off:

R-F estimates are robust to identification conditions on the structural form. However if at least one equation is overidentified, we can obtain more efficient estimates of the reduced form equations by means of structural estimates (though asymptotic variances are complicated).

2.3 Identification with cross equation restrictions

Identification can be helped with cross equation restrictions. The analysis is here similar to the previous one for a single equation.

Suppose again that the information on $\boldsymbol{\beta}$ is via a set of **homogeneous linear restrictions**:

$$\underset{J\times G(G+M)}{\mathbf{R}}\boldsymbol{\beta} = \mathbf{0}; \quad \text{where} \quad \underset{G(G+M)\times 1}{\boldsymbol{\beta}} := \text{vec}[\mathbf{B}] = \begin{pmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_G \end{pmatrix}$$

where J is the number of known (additional) restrictions on the vector $\boldsymbol{\beta}$ and is the rank of $\mathbf{R}$ (no redundant restrictions).

■ The **system rank condition** for identification of $\mathbf{B}$ is

$$\text{rank}\left[\mathbf{R}\left(\mathbf{I}_G \otimes \mathbf{B}\right)\right] = G\left(G-1\right)$$

taking into account the G normalization restrictions (see Ruud, 2000).

If there are only single equation restrictions, then $\mathbf{R}$ is block diagonal and we recover the single equation analysis.

■ The **system order condition** is

$$J \geq G\left(G-1\right).$$

EX. 2 (SEM: Cross equation restrictions.) *For a two equation system*

$$y_1 = \gamma_{12} y_2 + z_1 \delta_{11} + z_2 \delta_{12} + z_3 \delta_{13} + u_1 \qquad (2.10)$$
$$y_2 = \gamma_{21} y_1 + z_1 \delta_{21} + z_2 \delta_{22} + u_2 \qquad (2.11)$$

where the z_g *are orthogonal with the* u_h. *Equation (3.10) is unidentified without further restriction, and equation (3.11) is just identified iff* $\delta_{13} \neq 0$.

If we set $\delta_{12} = \delta_{22}$, *then since,* δ_{22} *is identified in the second equation,* δ_{12} *is also identified, and we can treat it as known. Then*

$$y_1 - z_2 \delta_{12} = \gamma_{12} y_2 + z_1 \delta_{11} + z_3 \delta_{13} + u_1.$$

Because now z_2 *is excluded on the rhs, we can use it as an instrument for* y_2 *(as long as* z_2 *appears in the reduced form of* y_2, *which is guaranteed by* $\delta_{22} \neq 0$, *so* $\delta_{12} \neq 0$*), identifying then (3.10).*

■ *Note that*

$$\boldsymbol{\beta} = (-1, \gamma_{12}, \delta_{11}, \delta_{12}, \delta_{13}, \gamma_{21}, -1, \delta_{21}, \delta_{22}, \delta_{23})'$$

and the matrix $\mathbf{R}$ *setting the* $J = 2$ *restrictions (*$\delta_{23} = 0$, $\delta_{12} = \delta_{22}$*) on the* $G(G+M) = 2(2+3) = 10$ *parameters is*

$$\mathbf{R} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -1 & \gamma_{11} \\ \gamma_{12} & -1 \\ \delta_{11} & \delta_{21} \\ \delta_{12} & \delta_{22} \\ \delta_{13} & \delta_{23} \end{pmatrix},$$

noting the two normalization restrictions. Then it is easy to check that

$$\begin{aligned} \mathbf{R}(\mathbf{I}_2 \otimes \mathbf{B}) &= \mathbf{R} \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{B} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \delta_{13} & \delta_{23} \\ \delta_{12} & \delta_{22} & -\delta_{12} & -\delta_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \delta_{13} & 0 \\ \delta_{12} & \delta_{12} & -\delta_{12} & -\delta_{12} \end{pmatrix} \end{aligned}$$

which is of rank $G(G-1) = 2$ *if* $\delta_{12} \neq 0$ *and* $\delta_{13} \neq 0$. □

2.4 Identification with covariance restrictions

Some times it might be interesting to set restrictions on the covariance matrix $\boldsymbol{\Sigma}$ of the structural errors. Typically these are zero covariance conditions that can help to identify structural coefficients.

EX. 3 (SEM: Covariance restrictions) *For a two equation system*

$$y_1 = \gamma_{12} y_2 + z_1 \delta_{11} + z_3 \delta_{13} + u_1 \tag{2.12}$$

$$y_2 = \gamma_{21} y_1 + z_1 \delta_{21} + z_2 \delta_{22} + z_3 \delta_{23} + u_2 \tag{2.13}$$

Equation (3.12) is just identified if $\delta_{22} \neq 0$. Equation (3.13) is unidentified with the information given.

Suppose that $\boldsymbol{\Sigma}$ is diagonal,

$$\mathbb{E}(u_1 u_2) = 0.$$

Since (3.12) is identified, we can treat $\gamma_{12}, \delta_{11}, \delta_{13}$ as known to study the identifiability of (3.13). Then u_1 can be treated also as known, and used as a further instrument in (3.13), since it is orthogonal with u_2, but correlated with y_1. □

Example 1 (SEM: Recursive systems.) *Consider the system*

$$\begin{aligned} y_1 &= \mathbf{z}'\boldsymbol{\delta}_1 + u_1 \\ y_2 &= \gamma_{21} y_1 + \mathbf{z}'\boldsymbol{\delta}_2 + u_2 \\ \vdots & \quad \vdots \\ y_G &= \gamma_{G1} y_1 + \cdots + \gamma_{G,G-1} y_{G-1} + \mathbf{z}'\boldsymbol{\delta}_G + u \end{aligned}$$

It is recursive because only the endogenous variables determined in previous equations appear in each successive equation. The first equation is identified and can be estimated by OLS. Without further restrictions the remaining equations are not identified. But we can set

$$\mathbb{E}(u_g u_h) = 0, \quad g \neq h = 1, \ldots, G.$$

This implies that $\boldsymbol{\Sigma}$ is diagonal. Then y_1 in the second equation is uncorrelated with u_2, and successively u_g is uncorrelated with $y_1, \ldots, y_{g-1}$. Therefore each equation can be consistently estimated by OLS. □

2.5 SEMs nonlinear in endogenous variables

Consider the supply and demand system

$$\begin{aligned}\log q &= \gamma_{12}\log p + \gamma_{13}\log^2 p + \delta_{11}z_1 + u_1\\ \log q &= \gamma_{21}\log p + \delta_{22}z_2 + u_1\\ \mathbb{E}(u_1|\mathbf{z}) &= \mathbb{E}(u_2|\mathbf{z}) = 0, \quad \mathbf{z} := (z_1, z_2)'\end{aligned}$$

where the first equation is the Supply equation and the second is the Demand equation. This system is linear in the parameters, but nonlinear in the endogenous variables, since we have the quadratic term $\log^2 p$, not because the transformations $\log p$ or $\log q$, since the system is equivalent to

$$y_1 = \gamma_{12}y_2 + \gamma_{13}y_2^2 + \delta_{11}z_1 + u_1 \tag{2.14}$$

$$y_1 = \gamma_{21}y_2 + \delta_{22}z_2 + u_1. \tag{2.15}$$

The reduced form for y_2 is nonlinear in the variables $\mathbf{z}$ and $\mathbf{u}$, so is $\mathbb{E}(y_2|\mathbf{z})$, for $\gamma_{13} \neq 0$, or $\mathbb{E}(y_2^2|\mathbf{z})$ (for any γ_{13}).

■ One approach is to suppose that the endogenous variables appearing on the rhs are not effectively the same and set $y_3 = y_2^2$:

$$y_1 = \gamma_{12}y_2 + \gamma_{13}y_3 + \delta_{11}z_1 + u_1. \tag{2.16}$$

This is OK as far rank and order conditions are satisfied. The problem is how these new variables are determined, since this equation has two endogenous variables and only one instrument available for identification, z_2.

• For the linear the model, $\gamma_{13} = 0$, and the R-F $y_2 = \pi_{21}z_1 + \pi_{22}z_2 + v_2$, $\mathbb{E}(v_2|\mathbf{z}) = 0$,

$$\mathbb{E}(y_2^2|\mathbf{z}) = \pi_{21}^2 z_1^2 + \pi_{22}^2 z_2^2 + 2\pi_{21}\pi_{22}z_1z_2 + \mathbb{E}(v_2^2|\mathbf{z}).$$

These are the natural instruments for $y_3 = y_2^2$, since we would expect that z_1^2, z_2^2, z_1z_2 are correlated with y_2^2.

• When $\gamma_{13} \neq 0$, we may wish to add more instruments, and in practice we can set

$$y_2^2 = \pi_{31}z_1 + \pi_{32}z_2 + \pi_{33}z_1^2 + \pi_{34}z_2^2 + \pi_{35}z_1z_2 + v_3$$

where v_3 is uncorrelated with $z_1, z_2, z_1^2, z_2^2, z_1z_2$. Now we can study the system with the usual rank condition adding this equation to the original system.

∗ However this procedure is equivalent to study the rank condition of the first two equations in the reduced system (3.16) and (3.15), without the equation for $y_3 = y_2^2$ and setting $G = 2$ (Fisher, 1965).

■ This **rank condition** *is not necessary* however. In the system

$$\begin{aligned} y_1 &= \gamma_{12}y_2 + \gamma_{13}y_2^2 + \delta_{11}z_1 + \delta_{12}z_2 + u_1 \\ y_2 &= \gamma_{21}y_1 + \delta_{22}z_2 + u_1 \end{aligned}$$

the first equation does not satisfy the order, neither the rank conditions.

• If $\gamma_{13} \neq 0$ and $\gamma_{21} \neq 0$, then $\mathbb{E}(y_2|\mathbf{z})$ is a nonlinear function of $\mathbf{z}$, so $z_1, z_2, z_1^2, z_2^2, z_1z_2$ all appear in the linear projections of y_2 and y_2^2 and therefore z_1^2, z_2^2, z_1z_2 are valid instruments for y_2 and y_2^2 in the first equation.

• But if $\gamma_{13} = 0$, this argument is not valid and the first equation is not identified.

This system is generally called *poorly identified*, because its possible identification relies on a nonlinearity (and the hypothesis $H_0 : \gamma_{13} = 0$ cannot be tested using estimates of the structural equation, since this is not identified under H_0.) For this reason is interesting to check if a linear version of the model is identified, as were equation (3.14).

■ For **estimation** of such nonlinear models in endogenous variables we would use in the instruments list the squares and cross products of all the exogenous variables. In general, adding more IV is not problematic, since asymptotically we can only improve, though finite sample performance can be affected.

Each equation can be estimated by 2SLS, or by general system methods. General chi-square estimates are preferred to 3SLS since heteroskedasticity cannot be ruled out in this framework. Furthermore, exact formulas should be used, because e.g. $\mathbb{L}_n(y_2|\mathbf{z})^2 \neq \mathbb{L}_n(y_2^2|\mathbf{z})$. Which is the one to be used?

RECOMMENDED READINGS: Wooldridge (2002, Ch. 9). Ruud (2000, Ch. 26.3-5), Mittelhammer (2000, Ch. 17).

RELATED TOPICS: FIML-LIML (Mittelhammer, 17.3; Davidson, 13.2-13.3)

Problem Set 2

1. Consider the following three-equations system

$$\begin{aligned} y_1 &= \gamma_{12} y_2 + z_1 \delta_{11} + z_2 \delta_{12} + z_3 \delta_{13} + u_1 \\ y_1 &= \gamma_{22} y_2 + \gamma_{23} y_3 + z_1 \delta_{21} + u_2 \\ y_3 &= z_1 \delta_{31} + z_2 \delta_{32} + z_3 \delta_{33} + u_3, \end{aligned}$$

   where $z_1 = 1$, $\mathbb{E}[u_g] = 0$ and each z_j is uncorrelated with each u_g. The first two equations can be considered as demand and supply equations, where the supply depends on a possibly endogenous variable y_3 (e.g. wage costs) that might be correlated with u_2. For example, u_2 might contain managerial quality.

   Show that a well-defined reduced form exists as long as $\gamma_{12} \neq \gamma_{22}$.

   Allowing for the structural errors to be arbitrarily correlated, determine which of these equations is identified and investigate the rank and order conditions.

2. Check the order and rank condition for the three equations in the following system:

$$\begin{aligned} y_1 &= \gamma_{12} y_2 + \gamma_{13} y_3 + z_1 \delta_{11} + z_3 \delta_{13} + u_1 \\ y_2 &= \gamma_{21} y_1 + z_2 \delta_{22} + u_2 \\ y_3 &= z_1 \delta_{31} + z_2 \delta_{32} + z_3 \delta_{33} + z_4 \delta_{34} + u_3 \end{aligned}$$

   where $z_1 = 1$, $\mathbb{E}[u_g] = 0$, and the z_g are uncorrelated with the u_h.

3. Consider the two linear simultaneous equations $(G = 2)$ with two exogeneous variables $(K = 2)$,

$$\begin{aligned} y_1 \gamma_{11} + y_2 \gamma_{12} + z_1 \delta_{11} + z_2 \delta_{12} &= u_1 \\ y_1 \gamma_{21} + y_2 \gamma_{22} + z_1 \delta_{21} + z_2 \delta_{22} &= u_2 \end{aligned}$$

   where, $\mathbf{u} = (u_1, u_2)'$,

$$E[\mathbf{uu}'] = \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{21} \\ \gamma_{12} & \gamma_{22} \end{bmatrix}.$$

(a) Using the standard normalization, write the general form of the order and rank conditions for single equation identification and for system identification. Then, stating the restrictions on the system parameters, check the identification of the above system in the following cases.

(b) SUR model.

(c) A different exogeneous variable is omitted from each structural equation.

(d) The variable z_2 does not appear in the system.

(e) Neither z_1 nor z_2 appear in the first equation.

(f) Γ is constrained to be symmetric and the coefficient in z_1 is the same in both equations.

(g) Γ is constrained to be lower triangular (with diagonal elements equal to 1) and Σ is diagonal. In this case explain how you would estimate the structural form parameters from the estimation of the reduced form. Are these estimates efficient in general? And if you further assume the restrictions on (a)?

(h) Which is your recommended estimation method in each case?

4. The following two-equation model contains an interaction between an endogenous and exogenous variable: Check the order and rank condition for the two equations in the following system:

$$\begin{aligned} y_1 &= \delta_{10} + \gamma_{12} y_2 + \gamma_{13} y_2 z_1 + z_1 \delta_{11} + z_2 \delta_{12} + u_1 \\ y_2 &= \delta_{20} + \gamma_{21} y_1 + z_1 \delta_{21} + z_3 \delta_{23} + u_2 \end{aligned}$$

where the z_g are orthogonal with the u_h.

(a) Suppose that $\gamma_{13} = 0$, so the model is a linear SEM. Discuss identification of each equation.

(b) For any value of γ_{13}, find the reduced form for y_1 (assuming it exists) in terms of the z_j, the u_g and the parameters.

(c) Assuming $\mathbb{E}[u_g|\mathbf{z}] = 0$, find $\mathbb{E}[y_1|\mathbf{z}]$.

(d) Show that, under the conditions in part (a), the model is identified regardless of the value of γ_{13}.

(e) Suggest a 2SLS procedure for estimating the first equation.

(f) Define a matrix of instruments suitable for 3SLS estimation.

(g) Suppose that $\delta_{23} = 0$, but we also know that $\gamma_{13} \neq 0$. Can the parameters in the first equation be consistently estimated? If so, how? Can $H_0 : \gamma_{13} = 0$ be tested?

5. Assume that wage and alcohol consumption are determined by the system

$$\begin{aligned} wage &= \gamma_{12} alcohol + \gamma_{13} educ + \mathbf{z}'_{(1)} \boldsymbol{\delta}_{(1)} + u_1 \\ alcohol &= \gamma_{21} wage + \gamma_{23} educ + \mathbf{z}'_{(2)} \boldsymbol{\delta}_{(2)} + u_2 \\ educ &= \mathbf{z}'_{(3)} \boldsymbol{\delta}_{(3)} + u_3. \end{aligned}$$

The third equation is a Reduced Form for years of education. $\mathbf{z}_{(1)}$ contains a constant, experience, gender, marital status, and amount of job training. $\mathbf{z}_{(2)}$ contains a constant, experience, gender, marital status and local prices of alcoholic beverages. $\mathbf{z}_{(3)}$ can contain elements in $\mathbf{z}_{(1)}$ and $\mathbf{z}_{(2)}$ and exogenous factors affecting education (e.g. distance to nearest college at age 16). Let $\mathbf{z}$ denote the vector containing all nonredundant elements in $\mathbf{z}_{(1)}, \mathbf{z}_{(2)}, \mathbf{z}_{(3)}$. Assume $\mathbf{z}$ is uncorrelated with each u_1, u_2, u_3 and assume that *educ* is uncorrelated with u_2, but *educ* might be correlated with u_1.

(a) When does the order condition hold for the first equation?

(b) Describe a single-equation procedure to estimate the first equation.

(c) Define the matrix of instruments for system estimation of all three equations.

(d) In a system procedure, how should you chose $\mathbf{z}_{(3)}$ to make the analysis as robust as possible to factors appearing in the reduced form for *educ*?

6. Consider the following two-equation nonlinear-SEM

$$\begin{aligned} y_1 &= \delta_{10} + \gamma_{12} y_2 + \gamma_{13} y_2^2 + \mathbf{z}'_1 \boldsymbol{\delta}_1 + u_1 \\ y_2 &= \delta_{20} + \gamma_{21} y_1 + \mathbf{z}'_2 \boldsymbol{\delta}_2 + u_2 \end{aligned}$$

where u_1 and u_2 have zero means conditional on all exogenous variables $\mathbf{z}$. Assume that both equations are identified when $\gamma_{13} = 0$.

(a) When $\gamma_{13} = 0$, $\mathbb{E}[y_2|\mathbf{z}] = \pi_{20} + \mathbf{z}'\boldsymbol{\pi}_2$. What is $\mathbb{E}[y_2^2|\mathbf{z}]$ under homoskedasticity assumptions for u_1 and u_2?

(b) Use part (a) to find $\mathbb{E}[y_1|\mathbf{z}]$ when $\gamma_{13} = 0$.

(c) Use part (b) to argue that, when $\gamma_{13} = 0$, a regression of the first equation using $\mathbb{L}_n[y_2|\mathbf{z}]$ and $\mathbb{L}_n[y_2|\mathbf{z}]^2$ in place of y_2 and y_2^2 produces consistent estimates of the parameters, including γ_{13}. Can we always expect this result?

(d) If u_1 and u_2 have constant variances conditional on $\mathbf{z}$, and $\gamma_{13} = 0$, show that the optimal IV for estimating the first equation are $\left\{1, \mathbf{z}, \mathbb{E}\left[y_2|\mathbf{z}\right]^2\right\}$.

7. Use data in MROZ.RAW to estimate a labour supply function for working, married women, and a second equation specified as a wage offer function, with equilibrium conditions imposed,

$$\begin{aligned} hours &= \gamma_{12}\log(wage) + \delta_{10} + \delta_{11}educ + \delta_{12}age \\ &\quad +\delta_{13}kidslt6 + \delta_{14}kidsge6 + \delta_{15}mwifeinc + u_1 \\ \log(wage) &= \gamma_{21}hours + \delta_{20} + \delta_{21}educ + \delta_{22}exper + \delta_{23}exper^2 + u_2 \end{aligned}$$

where kidstl6 is number of children less than 6, kidsge6 is the number of children between 6 and 18 and mwifeinc is income other than the woman's labor income. We assume that u_1 and u_2 have zero mean conditional on *educ*, *age*, *kidslt*6, *kidsge*6, *nwifeinc* and *exper*.

(a) Study the identification of the labor supply function when *exper* and *exper*^{2} have not direct effect on current annual hours. Is this equation no, just or over-identified?

(b) Estimate this labor supply function by OLS (ignoring the endogeneity of log(*wage*)) and by 2SLS using as IV all exogenous variables in both equations. Compare the results, in particular the sign of γ_{12}. Test the significance of the possible overidentifying restrictions.

(c) Is the wage offer function identified with the exclusion restrictions imposed?

Estimate a reduced form for *hours* (including all exogenous variables excluded from this equation) to check whether these are valid IV.

Estimate then by 2SLS. Is $\hat{\gamma}_{21}$ significative?

(d) Rewrite the second equation as a demand function, with *hours* as dependent variable. Repeat the steps of the previous point. Is 2SLS estimation very different form OLS? Why you would expect this? What are the consequences on identification?

(e) Estimate the two original equations by system procedures using the same common list of instruments (use the command INST in Eviews for setting this list).

Compare the 2SLS and the 3SLS estimates. Which additional information are using the latter? In which case would the two methods produce the same estimates?

Use GMM estimation. What is the difference with respect to 3SLS?

(f) Assume that *educ* is endogenous in the second equation, but exogenous in the first. Write then a three-equation system using different instruments for different equations, where *motheduc*, *fatheduc*, and *huseduc* are assumed exogenous in the two original equations (use the command @ in Eviews to specify the list of IV for each equation).